# NEGATIVE LATIN SQUARE TYPE PARTIAL DIFFERENCE SETS FROM NONABELIAN GROUPS AND THEIR PRODUCT CONSTRUCTIONS

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## BACKGROUND AND MOTIVATION

- In 2009, Polhill introduced product constructions for Negative Latin Square-type partial difference sets (PDS), but these were restricted to abelian 2-groups that had a specific (16, 5, 0, 2) PDS partition.
- Intelligent computer search by Andrew Brady published in 2024 identified 8 candidate groups of order 16 (regardless of commutativity) that contain a (16, 5, 0, 2) PDS.
- Brady's search confirmed the existence of a PDS, but it did not determine if these groups admitted the necessary partition required to use the 2025 theorem.
- Research published in 2025 by Davis, Polhill, et. al, proved Polhill's constructions are valid if the partition parameters match, regardless of whether the group is commutative.
- For the first time, we show a product construction on non-abelian group candidates to determine which of them admit the (16, 5, 0, 2) PDS partition, thereby extending Polhill's construction to non-abelian groups.

## DIFFERENCE SETS

- A (v, k, λ) difference set D is a subset of a group G of order v such that |G| = v, |D| = k, and the multiset of pairwise inverses $\Delta = \{d_1 d_2^{-1} \mid d_1, d_2 \in D\}$, such that $d_1 \neq d_2$ contains every nonidentity element of G exactly λ times.
- A (v, k, λ, μ) partial difference set D is a subset of a group G of order v where |G| = v, |D| = k, and the multiset of pairwise inverses $\Delta = \{d_1 d_2^{-1} \mid d_1, d_2 \in D\}$, such that $d_1 \neq d_2$ contains every non-identity element of D exactly λ times and every non-identity element of G \ D exactly μ times.
- To non-mathematicians, difference sets are important as they are used to construct error-correcting codes (like those in CDs and digital communication) to ensure data is transmitted accurately over noisy signals. They also form the basis of signal design for systems like sonar and radar to help clearly distinguish targets by reducing interference.
- Difference sets can construct finite geometries, such as the Fano Plane. It is the smallest possible projective plane, having 7 points and 7 lines. A difference set with parameters (7, 3, 1) (defining G = $Z_7$) can be used to construct the plane as seen in Figure 1.

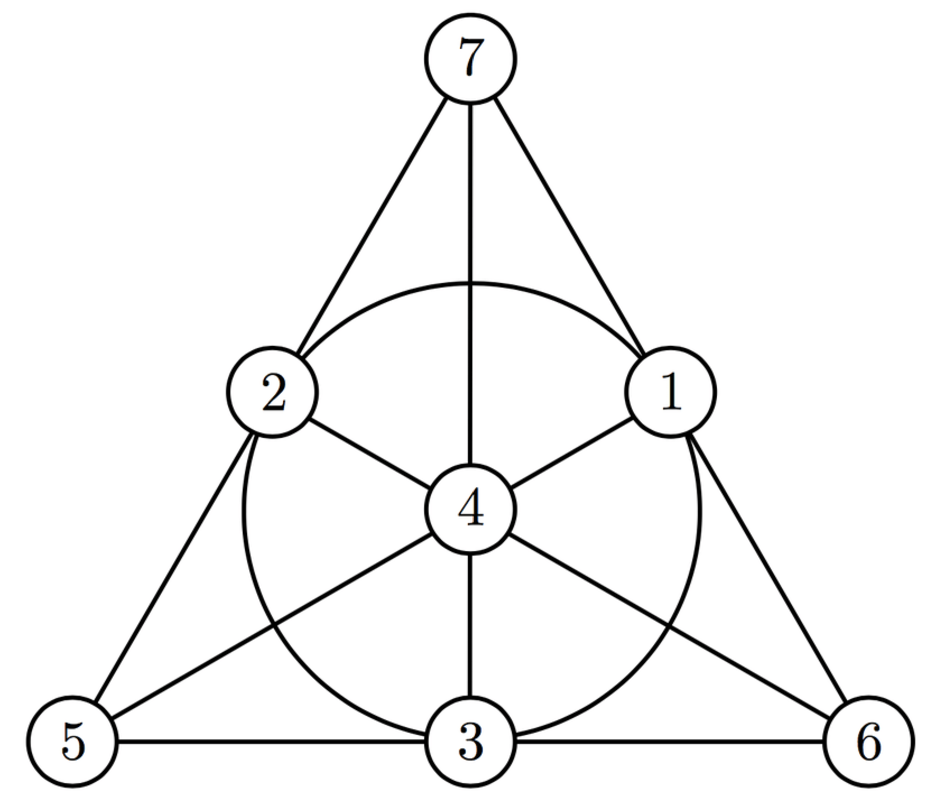
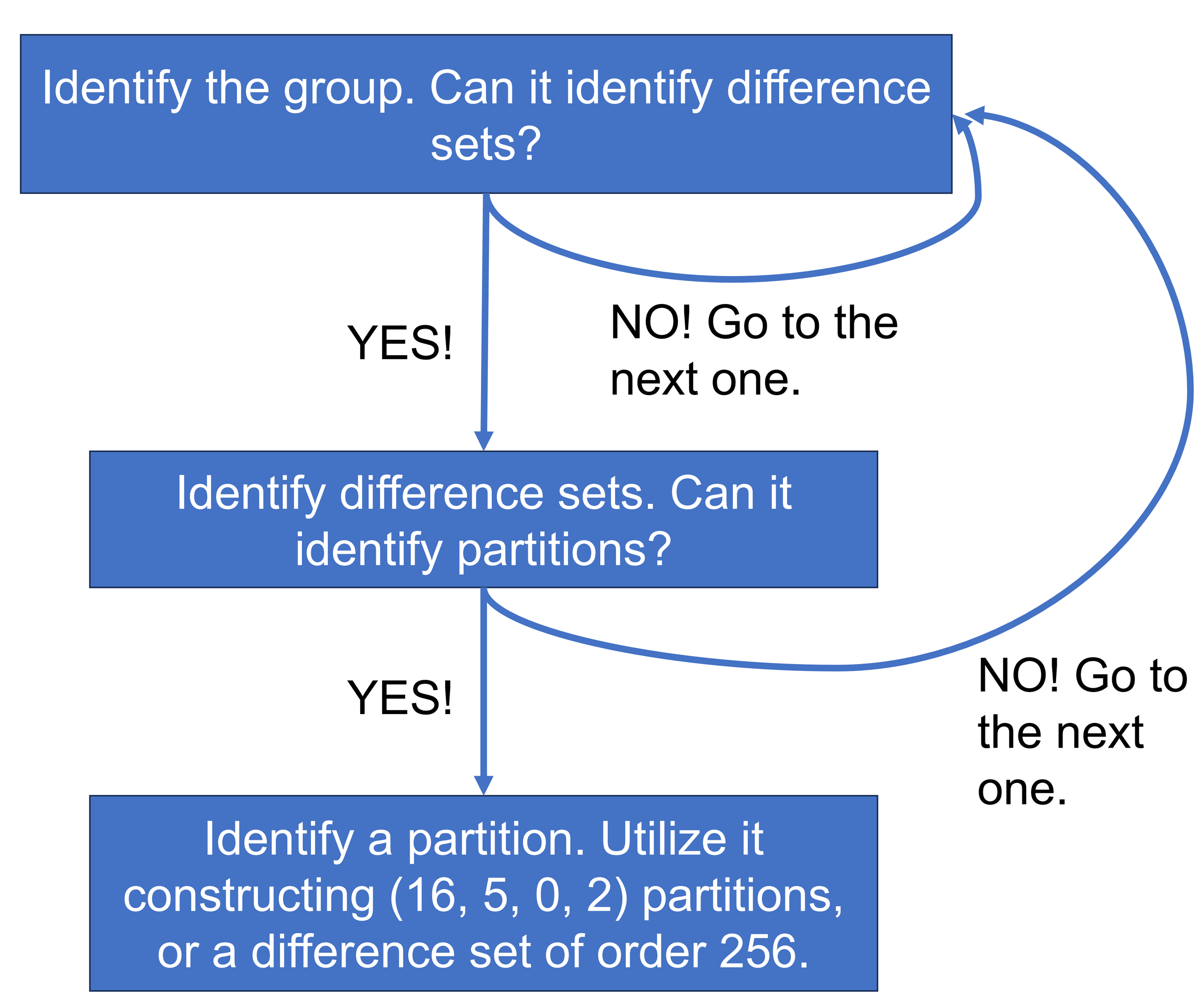



Fig. 1: Geometric depiction of the Fano plane.

## METHODS

- We conducted a computational search using the GAP (Groups, Algorithms, and Programming) system, common in algebraic research.
- Our goal was to determine which of these 8 groups admit a PDS partition, a specific structure required by Polhill's 2009 constructions.
- The 8 groups of order 16 that Andrew Brady's 2024 work identified as containing at least one (16, 5, 0, 2) partial difference set (PDS) were our partition search interest.
- We defined a successful partition as a collection of three disjoint (16, 5, 0, 2) partial difference sets whose union is the set of all 15 non-identity elements of the group.
- Adapted scripts, originally developed by a correspondence Dr. Ken Smith, to perform an exhaustive search. Iterates through all 14 groups of order 16 and, for each, checks combinations of its subgroups to find sets that satisfy the required partition parameters.

## FLOWCHART FOR SEARCHING GROUPS OF ORDER 16



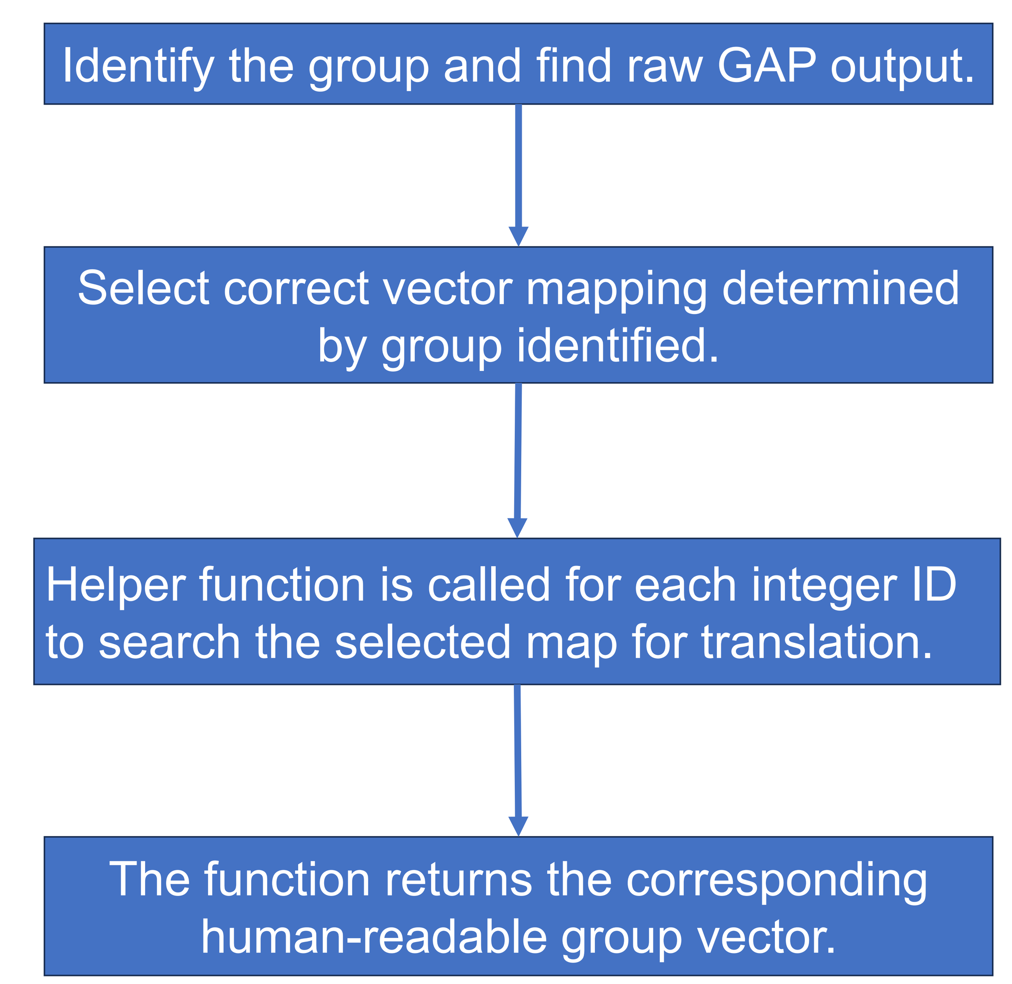

## GROUP PARTITION

Let G be a finite group of order v with identity elements and let G* = G \ {e} denote the set of non-identity elements. A partition of G* by partial difference sets, or simply a PDS partition of G, is a collection of m subsets $D = \{D_1, D_2, \ldots, D_m\}$ of G that satisfies the following conditions:

- Each subset $D_i \in D$ is a (v, k, λ, μ) partial difference set in G for the same set of parameters.
- The subsets are pairwise disjoint: $D_i \cap D_j = \emptyset$ for all $1 \leq i < j \leq m$.
- The union of the subsets is exactly G*, the set of all non-identity elements.

In our specific case, v = 16 and k = 5. The condition of m × k = v − 1 becomes m × 5 = 15, which forces m = 3. Our investigation is therefore to find which of the 8 groups identified by Brady admit such a partition into three disjoint (16, 5, 0, 2) partial difference sets.

These partitions are processed in GAP and translated to show these in terms of the actual vectors via input fed by a GAP script. Below is a flowchart description:




## PARTITIONS

- Following this process, we confirmed that exactly two non-abelian groups of order 16 admit the required (16, 5, 0, 2) partial difference set (PDS) partition.
- These two groups are:
  - $Z_2 \times D_4$ (The direct product of the cyclic group of order 2 and the dihedral group of order 8)
  - $(Z_2 \times Z_2) \rtimes Z_4$ (The semidirect product of $Z_2$ of the cyclic group of order 2 and the cyclic group of order 4)
- Crucially, this partition is identical in its parameters to the partition found in the abelian groups $Z_4 \times Z_4$ and $Z_2 \times Z_2 \times Z_2 \times Z_2$. This identity allows us to substitute these non-abelian groups into Polhill's original product constructions.
- These groups generate a new family of Negative Latin Square type PDSs in non-abelian product groups, extending the scope of Polhill's 2009 construction (described in next section).

## PRODUCT CONSTRUCTIONS

- Polhill's original 2009 product constructions relied on character theory, which hampers the potential for generalization to nonabelian groups.
- Ĝ x G', the newly defined product group, generates a new family of Negative Latin Square type partial difference sets of order 256 via a generalized product construction technique proposed by Davis, et. al (2025).
- Because the original templates allow for component groups of increasing size G' of order $2^{2s}$, our discovery effectively generates an infinite family of non-abelian Negative Latin Square type PDSs for orders $2^{2s+4}$ (this poster shows the base case).
- Substituting our groups into the templates generates two new Negative Latin Square type PDSs via non-abelian groups of order 256: a (256, 85, 24, 30) PDS and two disjoint (256, 51, 2, 12) PDSs.

## FUTURE IDEAS

- While our framework accounts for the base case of the family (order 256), future work involves explicitly constructing and computationally verifying the PDSs for higher orders (recursive case) to empirically demonstrate the infinite family property.
- Our current exhaustive search is computationally limited to small groups. An aspirational goal includes designing a local search-inspired algorithm (by adapting Brady's methods) to discover partitions in larger order where the number of groups that can be defined for that order expands rapidly.
- Construct SRGs from these new non-abelian sets and algorithmically test for isomorphism against known abelian versions.
- Polhill (2009) lists finding Negative Latin Square type PDSs in non-p-groups as a major open problem. A logical extension can be to apply our search methods to groups of non-prime-power orders to see if structures exist outside of p-groups.
- He also lists finding similar PDSs in p-groups for odd primes. The case when p=3 already has analogs provided to some of these constructions presented in the 2009 paper.

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